1

§0 Fixes from last time.

e.g.

R ✓ S ✓ T ×

$S = \mathbb{R}$, $a R b \iff |a-b| \le 1$

$1R2$, $2R3$, $1 \not R 3$

2

R ✓ S ✗ T ✓

$S = \mathbb{Z}$ $aRb \Leftrightarrow a \mid b$

$4 \mid 8$, $8 \nmid 4$

R ✗ S ✓ T ✓

$S = \mathbb{Z}$ $aRb \Leftrightarrow ab$ is odd

S: $aRb \Leftrightarrow ab$ odd $\Leftrightarrow ba$ odd $\Leftrightarrow bRa$

3.

$aRb,\ bRc \Rightarrow a, b, c$ are all odd

$\Rightarrow ac$ is odd

$\Rightarrow aRc$

$\{0,1,2,3\}$

odd

Even

$2R2$ $\quad 2 \cdot 2 = 4$ even not odd

prop
$S, T \Rightarrow R$

"pf:" $\quad xRy \xRightarrow{S} yRx \xRightarrow{T} xRx$ $\blacksquare$

$S = \{1, 2, 3\}$

$R = \{(1,1), (3,1), (1,3)\}$

Prop: If a relation is sym + trans and every element of S is relate to something then it is reflexive.

$a \mid b \Leftrightarrow b = a \cdot x$ $\qquad c = b \cdot y = a \cdot x \cdot y$

$b \mid c \Leftrightarrow c = b \cdot y$ $\qquad a \mid c$

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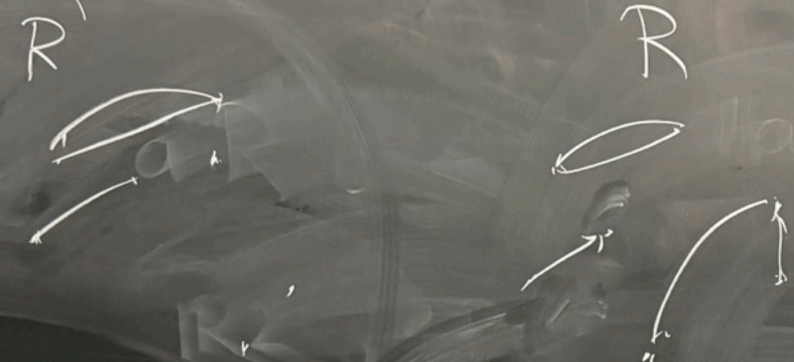
§1 Closures

<u>Defn:</u> A relation R <u>dominates</u> R' if $R' \subseteq R$ as a subset of $S \times S$.

<u>Visually:</u>



6.

e.g.) $S = \mathbb{Z}$, $aRb \Longleftrightarrow a < b$

$aRb \Longleftrightarrow a \leq b$

Defn: Let R be a relation on S, and $P \in \{$symm., reflex., trans.$\}$. The P-closure of R is the smallest relation dominating R which is P.

e.g.) $S = \mathbb{Z}$, $aRb \Longleftrightarrow a < b$, what is the reflexive closure? what is the symm closure? what is the trans closure?

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Step 1: Find guess R,

Step 2: R' is P and dominates R

Step 3: If R'' is P and dominates R, then $R'' \geq R'$.

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<u>Symm</u>: $aR'b \iff$ (either $a<b$ or $b<a$)

$$\Updownarrow$$

$$a \neq b$$

$aR'b \iff \begin{pmatrix} aRb \\ \text{or} \\ bRa \end{pmatrix}$

$bR'a \iff \begin{pmatrix} bRa \\ \text{or} \\ aRb \end{pmatrix}$

$R = \{(a,b)\} \subseteq S \times S$

$R' = \{(a,b) \in S \times S : (a,b) \in R \text{ or } (b,a) \in R\}$

9.

Prop: $\overset{R'}{\overset{\|}{\neq}}$ is symm. closure of $\overset{R}{\overset{\|}{<}}$.

Pf: First, observe that $\neq$ is symm: if $a \neq b$ then $b \neq a$, and dominates $<$: if $a < b$, $a \neq b$. Assume that R'' is a symm relation dominating $<$. We want to show that if $a \neq b$ then $a R'' b$. Either $a < b$ or $b < a$ and as R'' dominates $<$ $\Rightarrow$ $a R'' b$ or $b R'' a$. As R'' is symm in either case $a R'' b$. $\blacksquare$

$\forall R''$ dominates R and is symm $\Rightarrow$ R'' dominates R'

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Reflex: Reflexive closure of $a<b$ is $a \leq b$

Trans: $<$ $x>0$

Thm: If R is a relation, then a P-closure exists.

Lemma: If $\{R_i\}$ is any non-empty collection of relations which are P, then $R = \bigcap_i R_i$ is P.

$aR'b \Longleftrightarrow aR_ib$ for all i.

<u>Pf</u> (transitive): Suppose $aR'b$ and $bR'c$.

But $aR'b$ means $(a,b) \in R' = \bigcap_i R_i$, so

$(a,b) \in R_i$ for all i, i.e., aR_ib for all i.

Similarly, bR_ic for all i. Since R_i is trans,

aR_ic for all i, i.e., $(a,c) \bigcap R_i = R'$, so $aR'c$. $\blacksquare$

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<u>Pf of Thm</u>: Let $\{R_i\}$ be the set of all relations dominating R and which are P. Note, $S \times S \in \{R_i\}$.

Set $R' = \bigcap_i R_i$. By the lemma, R' is P and dominates R.

Assume R'' is P and dominates R. Then $R'' \in \{R_i\}$.

So, $R'' \supseteq \bigcap_i R_i = R'$, so R'' dominates R'. $\blacksquare$

<u>Note</u>: If R' and R'' are P-closures of R

$\Rightarrow R' \subseteq R''$, $R'' \subseteq R' \Rightarrow R' = R''$.

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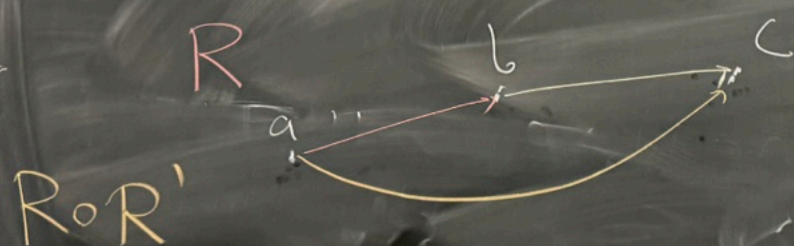
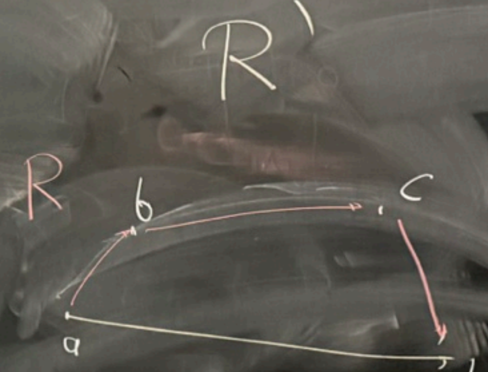
Def'n: For two relations R and R' are relations their **composition** is

$$R \circ R' = \{(a,c) \in S \times S \text{ s.t. } aRb \text{ and } bR'c\}$$

Visually:



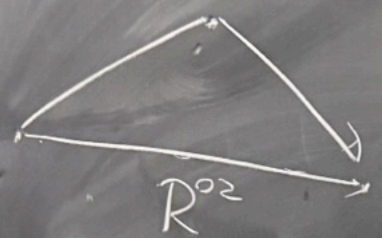


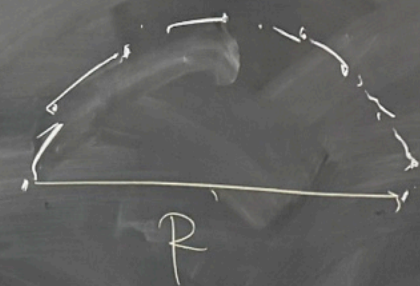
$$R^{\circ n} = R \circ R^{\circ(n-1)}$$

$R^{\circ 2}$

$a <^{\circ 2} d \iff a < (b < c) < d$

Prop: $R = R^{\circ 2}$ ($\Leftarrow$ $R = R^{\circ n}\ \forall n$) $\Longleftrightarrow$ R trans





Prop: The trans closure of R is $\bigcup_{n=1}^{\infty} R^{\circ n} = R'$

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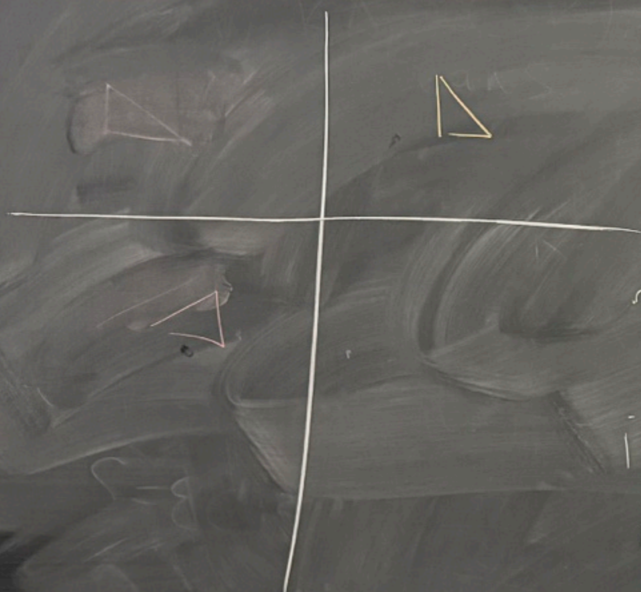
§2 Equivalence relations



__Defn:__ An __equivalence__ __relation__ is a relation which is symm, reflex., trans.

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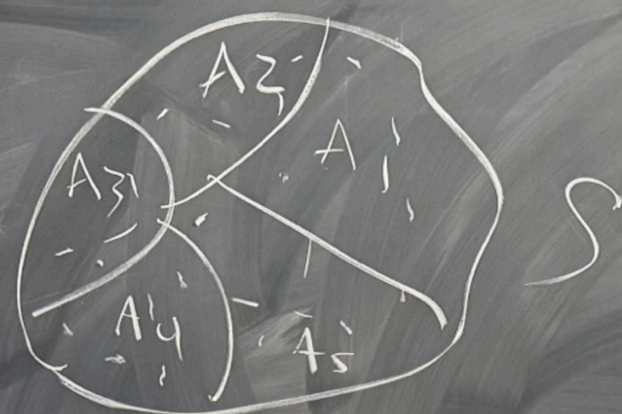
e.g.) $S = \{\text{animals}\}$

$aRb \iff a$ and b are same species.

Def'n: A partition of a set S is a collection $\{A_i\}$ of subsets of S s.t. $S = \bigcup_i A_i$, $A_i \cap A_j = \emptyset$ for $i \neq j$.

Visually:



Thm: Partitions "=" equivalence relations